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Multistep Guaranteed Cost Control of Linear Systems with Uncertain Parameters

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In many physical systems, an accurate knowledge of certain parameters is very difficult or very expensive to obtain. The designer of a remotely piloted vehicle (RPV) flight control system, for example, frequently has little data available regarding aerodynamic coefficients, due to a lack of wind tunnel tests. Based on the concept of guaranteed cost control, an algorithm has been developed to analyze the effect of parameter uncertainties on closed-loop system stability. A multistep extension of the guaranteed cost control method is developed for choosing constant feedback gains which result in stable closed-loop behavior for a range of parameter values. This technique has been applied to the design of a lateral autopilot for a rudderless RPV with uncertain aerodynamic coefficients.

Introduction

THE design of feedback controllers for systems with uncertain parameters has been a topic of interest to control system designers for many years. Parameter uncertainty can be dealt with in several ways. It can often be reduced substantially through extensive testing or through use of real-time or nonreal-time system identification techniques.[1,2] Alternately, parameter uncertainties may simply be accepted at their a priori levels, and a control system designed so as to be, in some sense, insensitive to parameter variations. It is the latter approach that is investigated in this paper. In particular, a linear, constant gain feedback controller is sought for a linear system with an uncertain system matrix, such that the closed-loop system behavior is acceptable for all values of the uncertain parameters within specified limits. Full-state feedback is assumed available.

Considerable effort has been expended in this area over the years, and a number of controller design methods have been devised. To date, no one technique has received widespread acceptance from control system designers. In Ref. 3, a comparative assessment of seven such methods was made in the context of wind load alleviation for the C-5A, with uncertainties assumed to exist in dynamic pressure, structural damping and frequency, and the stability derivative M_w. The techniques investigated were referred to as the additive noise design, the minimax design,[4] the multiplant design, the sensitivity vector augmentation design,[5] the state dependent noise design, the mismatch estimation design, and the uncertainty weighting design. Most of the methods were found to be at least somewhat burdensome computationally, and most did not produce control system designs judged to be significant improvements over a standard linear-quadratic synthesis design[6] which assumes precisely known parameters. The uncertainty weighting and minimax techniques were judged to be generally superior to the other approaches. In Ref. 7, the comparative assessment in Ref. 3 was extended to include an information matrix approach[8] and a finite-dimensional inverse method. The former method was found to perform favorably when compared with the minimax and uncertainty weighting methods.

Other methods proposed in recent years include the random variable method of Hadass and Powell,[9] continuous and discrete versions of the minimum expected cost method of Ly and Cannon, and Vinkler and Wood,[10-12] a related method due to Heath and Dillow,[13] and the guaranteed cost control method of Chang and Peng.[14] In the latter approach, controller feedback gains are determined by solving a matrix algebraic equation similar to the familiar Riccati equation, but with an additional term which bounds the effects of parameter uncertainties. Unlike some of the design techniques just referred to, the guaranteed cost control method is formulated to handle large parameter uncertainties, not just small perturbations about the nominal parameter values.

In this paper, the guaranteed cost control design method is re-examined. Certain optimality and stability properties of control systems designed by this technique are deduced. One drawback to the guaranteed cost control approach is its tendency to produce relatively large controller feedback gains, relatively large control effort, and overdamped dominant closed-loop poles. A technique is suggested here for avoiding this property. This technique involves constructing the controller in several steps, rather than in a single step as in Ref. 14.

This multistep form of the guaranteed cost control method is applied to two examples—a second-order system with two uncertain parameters, and a fifth-order lateral autopilot for a rudderless remotely piloted vehicle (RPV) with an uncertain aerodynamic coefficient $C_{n_{\delta_a}}$.

Guaranteed Cost Control of Linear Systems with Uncertain Parameters

The following assumptions are employed in the body of the paper:

1) System dynamics are described by the differential equation

$$\dot{x}(t) = A[q(t)]x(t) + Bu(t) \qquad (0 \leq t \leq t_f) \tag{1}$$

where $x(t)$ is the state vector $(n \times 1)$, $u(t)$ is the control vector $(m \times 1)$, $q(t)$ is the vector of uncertain parameters, referenced to their nominal values $(n' \times 1)$, $A[q(t)]$ is the

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open-loop dynamics matrix $(n\times n)$, B is the control distribution matrix $(n\times m)$, and t is the independent variable. The matrix A depends on time only implicitly through the uncertain parameter vector q, which may vary with time. The matrix B is constant and known precisely. Nominal design conditions are $q(t)\equiv 0$, $0\leq t\leq t_I$.

2) The vector $q(t)$ lies somewhere within a closed, bounded region $\Omega\subset R^{n'}$ (n'-dimensional real Cartesian space) for $0\leq t\leq t_I$. For simplicity, it is assumed that Ω is rectangular in shape and includes the origin, so that each component of q is bounded above and below as follows:

$$a_i\leq q_i\leq b_i \qquad a_i\leq 0 \qquad b_i\geq 0 \qquad (i=1,\ldots,n') \tag{2}$$

3) The structure of $A(q)$ is as follows:

$$A(q)=A_0+\sum_{i=1}^{n'} q_iA_i \tag{3}$$

where A_i, $i=0,\ldots,n'$, are constant $n\times n$ matrices, i.e., the uncertain parameters are assumed to enter A in a linear fashion.

4) (A_0,B) form a controllable pair.

5) Control system performance is characterized by a quadratic cost functional J, defined as follows:

$$J=\int_0^{t_I}(x^TQx+u^TRu)\mathrm{d}t \tag{4}$$

where Q is an $n\times n$ constant positive semidefinite symmetric matrix and R is an $m\times m$ constant positive definite symmetric matrix.

6) $(A_0,Q^{1/2})$ form an observable pair, where $Q^{1/2}$ denotes any square root of the matrix $Q[(Q^{1/2})^TQ^{1/2}=Q]$. (Construction of matrix square roots is discussed in Ref. 15, for example.)

7) All state variables are available for feedback.

The concept of guaranteed cost control can be introduced by means of the following theorem:

Theorem 1: Let $S(t)$ be the $n\times n$ symmetric matrix which satisfies the differential equation

$$\dot S(t)+S(t)A_0+A_0^TS(t)-S(t)BR^{-1}B^TS(t)+Q+U[S(t)]=0 \qquad (0\leq t\leq t_I) \tag{5}$$

subject to the boundary condition

$$S(t_I)=0 \tag{6}$$

where the matrix $U(S)$ is an upper bound on $\sum_{i=1}^{n'} q_i(SA_i+A_i^TS)$ in the sense that

$$x^TU(S)x\geq x^T\Big[\sum_{i=1}^{n'} q_i(SA_i+A_i^TS)\Big]x \tag{7}$$

for all $q\epsilon\Omega$ and $x\epsilon R^n$. Then the value of performance functional (4) achieved using the control law

$$u(t)=-C(t)x(t) \tag{8}$$

$$C(t)=R^{-1}B^TS(t) \tag{9}$$

is bounded above as follows:

$$J\leq x^T(0)S(0)x(0) \tag{10}$$

A proof of Theorem 1 is given in the Appendix. The matrix $S(t)$ is called a guaranteed cost matrix, and the control law specified by Eqs. (8) and (9), a guaranteed cost control law.

One convenient form of U is the following:

$$U=\sum_{i=1}^{n'} N_iE_iN_i^T \tag{11}$$

where N_i is the orthogonal transformation which diagonalizes the symmetric matrix $(SA_i+A_i^TS)$:

$$N_i^T(SA_i+A_i^TS)N_i=\Lambda_i \tag{12}$$

[Λ_i is diagonal and contains the eigenvalues of $SA_i+A_i^TS$, i.e., $(\Lambda_i)_{kk}=(\lambda_i)_k$, $(\Lambda_i)_{kj}=0$, $k\neq j$, where $(\lambda_i)_k$ is the kth eigenvalue of $SA_i+A_i^TS$]. E_i is defined by

$$(E_i)_{kk}=\begin{cases}a_i(\lambda_i)_k & [(\lambda_i)_k<0]\\ b_i(\lambda_i)_k & [(\lambda_i)_k\geq 0]\end{cases}$$

$$(E_i)_{kj}=0 \qquad (k\neq j) \tag{13}$$

U is, therefore, a function of S and A_i, a_i, and b_i, $i=1,\ldots,n'$. For notational simplicity, the functional dependence of U on quantities other than S is not explicitly indicated below. Note that if a_i and b_i, $i=1\ldots,n'$, are scaled by a positive constant ρ, U is scaled by ρ also. This fact will be of use later.

The expression for $U(S)$ given by Eqs. (11-13) (based on Ref. 14) is not the only upper bound consistent with Eq. (7). Other upper bounds are given in Refs. 16 and 17. Note that U is not explicitly time dependent, even if the uncertain parameter vector q is time-varying, by virtue of the definition of U as an upper bound.

If the domain of parameter uncertainty Ω consists of a single point (the origin), then $U[S(t)]\equiv 0$, and Eq. (5) reduces to the familiar Riccati equation. In this case, as $t_I-t\to\infty$, $S(t)$ will tend to a constant, positive-definite solution, under the controllability, observability, and positivity assumptions made previously.[18] If Ω is not a single point (i.e., U is nonzero), a positive-definite steady-state solution to Eq. (5), subject to boundary condition (6), may or may not exist. Some remarks on the existence of such steady-state solutions are made in Ref. 14.

Stability properties of guaranteed cost controllers are described by the following Corollary to Theorem 1.

Corollary: A guaranteed cost controller produces a closed-loop system which is globally asymptotically stable as $t_I-t\to\infty$ for all $q(t)\epsilon\Omega$.

Comments regarding the proof of the corollary are given in the Appendix.

The derivations and results to this point are basically abbreviated versions of those in Ref. 14. Some extensions of these results to situations in which the B matrix is uncertain, output feedback rather than full-state feedback is available, and the uncertain parameters of A are bounded by an n'-dimensional polygon rather than an n'-dimensional rectangle are discussed in Ref. 19.

In the remainder of this paper, it will be assumed, for simplicity, that the uncertain parameter vector q is either constant or else varies slowly enough with time that it may be treated as constant. Many of the results and procedures described later can be extended to handle arbitrary (but bounded) time variations of q. Further information is given in Ref. 19.

Certain stability and optimality properties of guaranteed cost controllers not discussed in Ref. 14 will now be presented in the form of a theorem:

Theorem 2: If the solution to the differential equation

$$\dot S(t)+S(t)A_0+A_0^TS(t)-S(t)BR^{-1}B^TS(t)+Q+\rho_dU[S(t)]=0 \qquad (0\leq t\leq t_I,\rho_d\geq 0) \tag{14}$$

subject to boundary condition (6) tends to a constant positive-definite value S_d as $t_f - t \to \infty$, the feedback control law

$$u(t) = -C_d x(t) \tag{15}$$

$$C_d = R^{-1} B^T S_d \tag{16}$$

is an optimal control law for all constant q such that $[A(q), B]$ form a controllable pair, $[A(q), Q^*(q)^{1/2}]$ form an observable pair, and

$$Q^*(q) = Q + \rho_d U(S_d) - \sum_{i=1}^{n'} q_i (S_d A_i + A_i^T S_d) \tag{17}$$

is positive-definite or -semidefinite. The associated state weighting matrix is $Q^*(q)$.

A proof of Theorem 2 is given in the Appendix. The following corollary to Theorem 2 can be deduced:

Corollary: Theorem 2 remains valid if the positivity condition is replaced by the condition that

$$\rho_a a_i \le q_i \le \rho_a b_i \qquad (i = 1, \ldots, n') \tag{18}$$

where ρ_a is the value of ρ which causes the quantity

$$Q_d(\rho) = Q + (\rho_d - \rho) U(S_d) \tag{19}$$

to change from positive-definite or -semidefinite to indefinite.

A proof of this corollary is given in the Appendix.

The controller optimality previously described is somewhat unusual in the sense that the controller is optimal with respect to different criteria (i.e., different values of Q^*) for different values of q. Nevertheless, for each value of q consistent with Eq. (18), the various desirable properties of an optimally designed control system described in Refs. 18, 20, 21 (e.g., global asymptotic stability, phase margin of at least 60 deg, infinite gain margin, tolerance of nonlinearities, etc.) hold. (It should be emphasized that these various optimality properties are achievable only when measurements of all state variables are available.) The corollary gives a lower bound on the region in parameter space for which the guaranteed cost controller produces a stable closed-loop system. Theorem 2 will give a less conservative lower bound, but is more difficult to implement. The actual region in parameter space for which the closed-loop system is stable is generally larger than the bounds calculated in Theorem 2 and its corollary. It is easily seen that ρ_a in the corollary is never smaller than ρ_d.

It is worth noting that the corollary to Theorem 2 may be used to determine stability boundaries in parameter space when some elements of q are assumed uncertain, with the remaining elements assumed known precisely. This is done by deleting terms corresponding to known parameters in the summation in Eq. (11).

Multistep Version of the Guaranteed Cost Control Design Technique

One of the drawbacks of the guaranteed cost control method for stabilizing systems with uncertain parameters is that it tends to result in overcontrolled behavior, i.e., large feedback gains and control effort, and frequently dominant closed-loop poles which are overdamped (see forthcoming examples). A technique, suggested for overcoming this difficulty, is as follows:

1) Select weighting matrices Q and R and the scalar $\mu, 0 < \mu \le 1$.

2) Determine the positive-definite solution S_0 to the following algebraic Riccati equation:

$$S_0 A_0 + A_0^T S_0 - S_0 B R^{-1} B^T S_0 + Q = 0 \tag{20}$$

by eigenvector decomposition.[22]

3) Evaluate the controller feedback gain matrix

$$C_0 = R^{-1} B^T S_0 \tag{21}$$

4) Evaluate $U(S_0)$ in accordance with Eqs. (11-13). Determine the value of ρ_0 for which the matrix

$$Q_0 = Q - \rho_0 U(S_0) \tag{22}$$

becomes indefinite. Set j equal to one.

5) Evaluate

$$F_{j-1} = A_0 - B C_{j-1} \tag{23}$$

6) Obtain the positive-definite solution S_j to the following equation:

$$S_j F_{j-1} + F_{j-1}^T S_j - S_j B R^{-1} B^T S_j + \mu Q + \rho_{j-1} U(S_j) = 0 \tag{24}$$

by an extension of Kleinman's method.[19,23]

7) Evaluate the feedback gain matrix

$$C_j = C_{j-1} + R^{-1} B^T S_j \tag{25}$$

8) Evaluate $U(S_j)$ in accordance with Eqs. (11-13). Determine the value of ρ_j for which the matrix

$$Q_j = \mu Q - (\rho_j - \rho_{j-1}) U(S_j) \tag{26}$$

becomes indefinite. If $\rho_j \ge 1$, a successful control system design has been achieved; if $\rho_j < 1$, but j is the maximum desired number of iterations, design process terminates unsuccessfully. Otherwise, increase j by one and go to step 5.

In step 1 of the preceding algorithm, the weighting matrices Q and R for the quadratic cost functional, Eq. (4), (with t_f infinite) are chosen by some suitable technique. They may be chosen by considering the sizes of state and control variables which are physically acceptable,[6] by model-matching techniques,[24] or by trial and error. The choice of μ will be commented upon in conjunction with step 6.

In steps 2 and 3, a set of feedback gains, C_0, is determined which results in a stable closed-loop system in the absence of parameter uncertainty, i.e., such that the eigenvalues of $F_0 = A_0 - BC_0$ have negative real parts. This closed-loop system is not only stable, but also possesses the desirable phase margin, gain margin, and disturbance insensitivity properties associated with an optimally designed system.[18,20,21]

In step 5, the closed-loop system matrix associated with nominal parameter values and the current set of feedback gains is evaluated.

In step 6, uncertainties in system parameters are taken into account in the controller design. μ may be chosen to be a small positive number, in order to place a relatively large penalty on additional control effort. As has been noted previously, one of the drawbacks of the design method in Ref. 14 is the relatively large control effort required. Note that if μQ had been used in place of Q in step 2, Eq. (20), the eigenvalues of F_0, though stable, would be reflections of the open-loop eigenvalues into the left half of the complex plane[25] for μ very small. If some open-loop eigenvalues were close to the imaginary axis, some closed-loop eigenvalues would be close also, for μ very small, and relatively small parameter variations could cause instability. For this reason, the system is first stabilized (for nominal parameter values) using a physically reasonable value of Q in Eq. (20), before trying to counter the problem of overcontrol associated with the guaranteed cost control method of handling parameter uncertainty.

In Ref. 14, equations resembling Eq. (24) are solved by backward integration until a steady-state solution is achieved, using Eq. (6) as a starting condition. In view of the fact that the steady-state solution to a Riccati equation can be found more efficiently by iterative techniques, such as the method of Kleinman,[23] than by integration, an iterative approach to solving Eq. (24) is likely to be superior to integration. The method of Kleinman can be extended without difficulty to solve this equation. Details are provided in Ref. 19.

In step 7, the controller feedback gain matrix is updated to include the incremental feedback gains calculated in the current iteration.

The motivation in step 8 follows from the corollary to Theorem 2, with ρ_j and ρ_{j-1} corresponding to ρ_a and ρ_d, respectively. In step 6, a guaranteed cost controller for q such that

$$\rho_{j-1}a_i \leq q_i \leq \rho_{j-1}b_i \qquad (i=1,\ldots,n') \tag{27}$$

is designed. In step 8, this controller is found to be an asymptotically stable controller for q such that

$$\rho_j a_i \leq q_i \leq \rho_j b_i \qquad (i=1,\ldots,n') \tag{28}$$

Once ρ_j has exceeded unity for any j, a controller design has been achieved which is stable and possesses guaranteed cost control properties for Eq. (2), which was the design objective. In addition, the controller design is an optimal one over at least a portion of this region.[19] Even though $\rho_j \geq \rho_{j-1}$ for all j, it may not be true that $\rho_j \geq 1$ for any j. ρ_j can tend toward a limiting value less than unity. This will happen in cases where constant feedback gains do not exist, which will stabilize the closed-loop system over the entire range of parameter uncertainty.

An alternative to step 8 that is sometimes useful is the following:

8′) If $\rho_{j-1} \geq 1$, a successful design has been achieved. Otherwise, let $\rho_j = \rho_{j-1} + \Delta\rho$, where $\Delta\rho$ is a positive quantity input to the routine either in advance or in an interactive mode as the computations proceed. Increase j by one and go to step 5.

Step 8 tends to produce fairly small changes in ρ from one iteration to the next when μ is small. The iterative process can be speeded up by using step 8′. Several other approaches to incrementing ρ from one iteration to the next are discussed in Ref. 19.

This multistep version of the guaranteed cost control technique, though requiring more computation than the guaranteed cost control method described in Ref. 14, avoids the unnecessarily tight control associated with that method.

Second-Order Example

Consider a second-order system of the form of Eq. (1) with

$$A(q) = \begin{bmatrix} 0 & 1 \\ -2+q_1 & 1+q_2 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{29}$$

where

$$-1 \leq q_1 \leq 1 \qquad -1 \leq q_2 \leq 1.5 \tag{30}$$

This problem was considered in Refs. 10-12. Controller designs achieved by several methods will be described. In each case, the weighting matrices Q and R were chosen to be

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad R = 10 \tag{31}$$

a) Controller design by linear-quadratic synthesis ignoring parameter uncertainty: Equation (20) is solved for its positive definite solution S_0 by the method of eigenvector decomposition, with

$$A_0 = \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix} \tag{32}$$

The controller gain matrix

$$C_0 = [C_{11} C_{12}] \tag{33}$$

is evaluated according to Eq. (21), and the results are tabulated in Table 1. In Table 2, the eigenvalues of the closed-loop system matrix $A(q) - BC_0$ are evaluated for five values of the uncertain parameter vector q:

$$q^0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad q^1 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \qquad q^2 = \begin{bmatrix} -1 \\ 1.5 \end{bmatrix}$$

$$q^3 = \begin{bmatrix} 1 \\ 1.5 \end{bmatrix} \qquad q^4 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{34}$$

(The region of parameter uncertainty Ω is a rectangle in R^2. q^0 corresponds to the nominal parameter vector; q^1, q^2, q^3, and q^4 are the corners of the rectangle.) Note that the closed-loop system is unstable for $q = q^2$ or q^3.

b) Controller design by the guaranteed cost control method of Chang and Peng: A positive-definite steady-state solution to Eqs. (5) and (6) is obtained by an extension of Kleinman's method, with $U(S)$ evaluated according to Eqs. (11-13). A_0 is given by Eq. (32), and A_1 and A_2 are given by

$$A_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{35}$$

Feedback gains and closed-loop eigenvalues are given in Tables 1 and 2. Note the relatively large feedback gains and the real, overdamped closed-loop eigenvalues.

c) Controller design by the multistep guaranteed cost control design method: Except for the choice of μ, steps 1-3 of this procedure have already been carried out in conjunction with design method a. μ was chosen to be 10^{-2}. The matrices A_0, A_1, and A_2 are the same as in design method b. In the first iteration of steps 4-8, step 8 yields $\rho_1 = 0.34$. Thus, a satisfactory design has been achieved for parameter variations limited to 34% of their maximum values. From Theorem 2, this design is not only stable over this range of parameter variations, but it is also an optimal design, with the associated desirable properties, for all parameter values within this range. As a matter of interest, ranges of optimality were also determined for q_1 uncertain and q_2 regarded as precisely known, and vice versa, at this stage in the iterative process. This was done by deleting one term in Eq. (11). For q_1 alone uncertain, ρ_1 was found to be greater than one. For q_2 alone uncertain, ρ_1 was found to be 0.55. Thus, the controller is optimal over 55% of the possible range of variation of q_2. A check of closed-loop eigenvalues indicated that the controller is stable over 71% of the full range.

In applying the multistep guaranteed cost control approach to this example, step 8′ was used initially instead of step 8, due to the small changes in ρ per iteration computed by the latter with $\mu = 10^{-2}$. Computer runs were made using three, four, and five iterations to complete the design process. Feedback gains and closed-loop eigenvalues for different

Table 1 Controller feedback gains determined by various design methods

Design method	C_{11}	C_{12}
a	0.03	2.07
b	1.36	6.42
c	0.30	3.44

Table 2 Closed-loop eigenvalues for various parameter values using feedback gains calculated by various methods

Design method	$q=q^0$	$q=q^1$	$q=q^2$	$q=q^3$	$q=q^4$
a	$-0.54\pm1.32j$	$-1.04\pm1.4j$	$+0.21\pm1.73j$	$+0.21\pm0.99j$	$-0.82, -1.26$
b	$-0.71, -4.71$	$-0.77, -5.65$	$-1.96\pm0.72j$	$-0.74, -3.18$	$-0.39, -6.03$
c	$-1.22\pm0.90j$	$-1.72\pm0.58j$	$-0.47\pm1.75j$	$-0.47\pm1.04j$	$-0.43, -3.01$

Table 3 Feedback gains and closed-loop eigenvalues for nominal parameter values achieved using the multistep guaranteed cost control method

Method of incrementing ρ	μ	Iterations of main algorithm	C_{11}	C_{12}	Closed-loop eigenvalues ($q=q^0$)
Step 8′	0.01	5	0.30	3.44	$-1.22\pm0.90j$
Step 8′	0.01	4	0.34	3.55	$-1.28\pm0.84j$
Step 8′	0.01	3	0.61	4.40	$-2.23, -1.17$
Step 8′	0.50	5	0.37	3.55	$-1.28\pm0.86j$
Step 8	0.50	9	0.36	3.40	$-1.20\pm0.96j$
Step 8	1.0	7	0.42	3.46	$-1.23\pm0.95j$

parameter values are given in Tables 1 and 2 for the five iteration case. Clearly, the multistep guaranteed cost approach avoids both the instability associated with the linear-quadratic design ignoring parameter uncertainty and the overdamped behavior associated with the original guaranteed cost control approach.

The feedback gains and closed-loop eigenvalues for nominal parameter values are compared in Table 3 for the three, four, and five iteration cases. Also included in Table 3 are results using step 8, rather than step 8′, for incrementing ρ. In these cases, the number of iterations required to complete the design is not an input quantity. It is determined automatically by the algorithm. Values of μ smaller than 0.5 were not used in conjunction with step 8, because of the relatively large number of iterations required to reach $\rho=1$.

Fifth-Order Lateral Autopilot for an RPV

Now, consider a system having the form of Eqs. (1-3), with

$$x^T=[v,p,r,\phi,\delta_a] \qquad u=\delta_{a_c} \qquad q=(C_{n_{\delta_a}}-1.99)/1.99 \tag{36}$$

where v is the component of vehicle velocity parallel to pitch axis, p is the vehicle roll rate, r the vehicle yaw rate, ϕ the vehicle roll angle, δ_a the aileron deflection, δ_{a_c} the commanded aileron deflection, $C_{n_{\delta_a}}$ the dimensionless partial derivative of moment about vehicle yaw axis with respect to aileron deflection, and

$$A_0=\begin{bmatrix} -0.85 & 25.47 & -979.5 & 32.14 & 0 \\ -0.339 & -8.789 & 1.765 & 0 & 59.89 \\ 0.021 & -0.547 & -1.407 & 0 & 6.477 \\ 0 & 1 & 0.0256 & 0 & 0 \\ 0 & 0 & 0 & 0 & -20 \end{bmatrix}$$

$$B=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 20 \end{bmatrix} \tag{37}$$

$$A_1=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.71 \\ 0 & 0 & 0 & 0 & 3.22 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{38}$$

$$-1.5\leq q\leq 0.5 \tag{39}$$

These perturbation equations describe the lateral dynamics of a particular RPV which has no rudder. The vehicle dynamics are described by the four state variables v, p, r, and ϕ. The fifth state variable, δ_a, is used to model aileron actuator dynamics as a first-order lag with a time constant of 0.05 s. The aerodynamic derivative $C_{n_{\delta_a}}$ is assumed to be an unknown constant within the range

$$-0.99\leq C_{n_{\delta_a}}\leq 2.99 \tag{40}$$

for this flight condition. Its nominal value is assumed to be 1.99. The following features are desired of the closed-loop system:

1) The closed-loop system must be stable for all $C_{n_{\delta_a}}$ consistent with Eq. (40).

2) The closed-loop poles corresponding to vehicle (as opposed to actuator) dynamics should be in the vicinity of -5.0, -0.2, and $-1.4\pm3.1j$ when $C_{n_{\delta_a}}$ takes on its nominal value.

3) The sensitivity of closed-loop pole locations to changes in $C_{n_{\delta_a}}$ should be as small as possible.

In accordance with requirement 2, the desired closed-loop behavior was modeled as

$$\dot{y}=Ly \qquad y=Hx \tag{41}$$

where

$$L=\begin{bmatrix} -2.6 & 0 & 2 & 0 \\ 0 & -2.8 & 0 & -11.57 \\ 2.88 & 0 & -2.6 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{42}$$

$$H=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{43}$$

The performance functional was chosen to be of the same form as Eq. (4) with[24]

$$Q=(HA_0-LH)^T Q_y (HA_0-LH) \tag{44}$$

$$Q_y=\mathrm{diag}(0.01, 0.1, 0.01, 0.1) \tag{45}$$

$$R=2500 \tag{46}$$

The controller design process was carried out using the three techniques described in conjunction with the second-order example. For method c, μ was chosen to be 10^{-4}, and step 8′ was used in lieu of step 8. Six iterations of the main algorithm were used. Controller gains for the three methods are tabulated in Table 4. Closed-loop eigenvalues for $C_{n_{\delta_a}}$ taking on its nominal value are given in Table 5. In Figs. 1 and 2, root locus plots of the three dominant closed-loop poles are presented for the three design methods, with $C_{n_{\delta_a}}$ varying between the limits specified in Eq. (40).

All three design methods produce stable closed-loop control systems when $C_{n_{\delta_a}}$ takes on its nominal value. When $C_{n_{\delta_a}}$ deviates from its nominal value, the standard linear-quadratic controller becomes unstable in the Dutch roll mode. The single-step and multistep guaranteed cost control approaches are stable for the full range of parameter uncertainty, but the former requires higher gains and a very fast actuator. The latter produces a closed-loop system which has most of the desired features, without the use of large feedback gains.

Table 4 Controller feedback gains for different design methods

Method	C_{11}	C_{12}	C_{13}	C_{14}	C_{15}
a	−0.0029	−0.084	1.80	0.013	0.34
b	−0.017	−0.10	2.84	−0.023	4.97
c	−0.0065	−0.075	1.83	0.012	1.24

Table 5 Closed-loop eigenvalues for nominal $C_{n_{\delta_a}}$ for different design methods

Method	Closed-loop eigenvalues ($C_{n_{\delta_a}}=1.99$)			
a	−19.74	−10.96	$-3.30\pm5.23j$	−0.62
b	−117.28	−9.90	$-1.57\pm6.51j$	−0.10
c	−41.28	−10.05	$-2.14\pm6.22j$	−0.29

The preceding discussion was based on the assumption that $C_{n_{\delta_a}}$ is an unknown constant within certain bounds. It should be noted that the multistep guaranteed cost control algorithm would have produced the same feedback gains if $C_{n_{\delta_a}}$ were assumed to vary in some arbitrary manner within this range. The resulting closed-loop system would be asymptotically stable for all such variations. Thus, for cases in which $C_{n_{\delta_a}}$ is a function of angle of attack, for example, which varies with time, this controller design would still be applicable (as long as the small perturbation model for the lateral dynamics is valid).

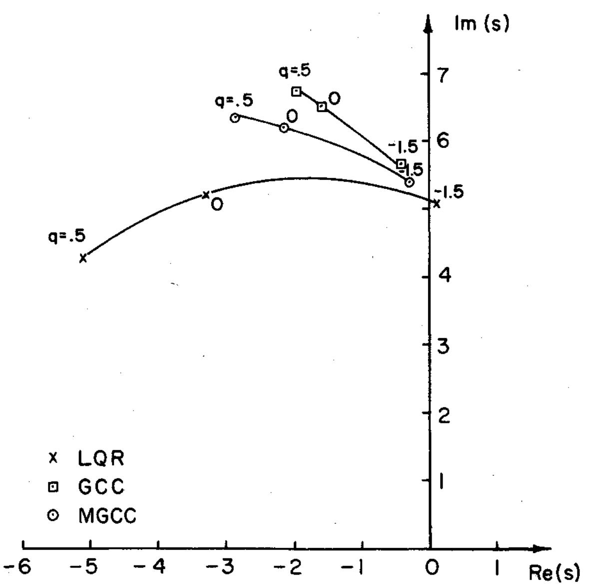

Fig. 1 Root locus plots of Dutch roll poles as functions of the uncertain parameter.

Conclusion

A new technique for designing constant gain feedback controllers for linear systems with large parameter uncertainty has been developed. The technique is derived from the guaranteed cost control method of Chang and Peng, but avoids the overcontrolled behavior often associated with that method. Overcontrolled behavior is avoided by performing the controller design in several steps, in which the assumed region of parameter uncertainty is gradually increased in size, rather than in a single step. The effects of parameter uncertainty on closed-loop system stability are easily observed as the controller design algorithm proceeds from one step to the next. The final design is guaranteed to be asymptotically stable for all parameter values within the specified region of uncertainty. (If the region of uncertainty is too large, it may not be possible to stabilize the system throughout this region with a single set of feedback gains. In this case, the algorithm will provide information about the contraction of the region required to allow such a design.) The design technique is applicable to systems with time-varying (but bounded), uncertain parameters, as well as to systems with constant uncertain parameters, although only the latter situation has been investigated extensively. In the constant parameter case, the controller design is optimal with respect to a quadratic performance functional, for all values of the uncertain parameters within some bounded region, the extent of which is easily determinable. The state weighting matrix, with respect to which the design is optimal, varies from point to point in parameter space.

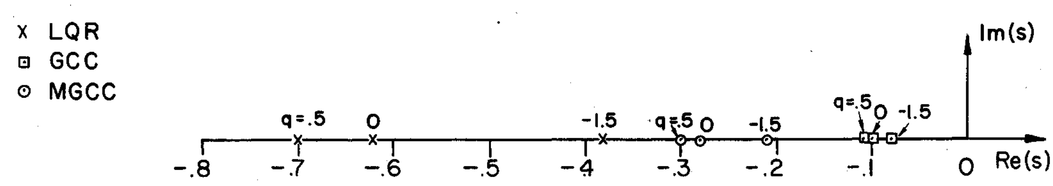

Fig. 2 Root locus plots of the spiral pole as a function of the uncertain parameter.

Two examples have been considered, one second-order with two uncertain parameters, the other fifth-order with one uncertain parameter, the latter example representing the lateral dynamics of an RPV. In both examples, the conventional linear-quadratic design for the nominal parameter values produced an unstable closed-loop system for some off-nominal parameter values. The single-step and multistep guaranteed cost control design techniques avoided this problem of instability. The latter required substantially less control effort than the former.

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Appendix

Proof of Theorem 1: Pre- and post-multiplication of Eq. (5) by $x^T(t)$ and $x(t)$ yields

$$x^T(t)\{\dot{S}(t)+S(t)A_0+A_0^TS(t)-S(t)BR^{-1}B^TS(t)+Q+U[S(t)]\}x(t)=0 \qquad (0\le t\le t_1, \forall x(t)\epsilon R^n) \quad \text{(A1)}$$

Use of Eqs. (3) and (7) yields

$$x^T(t)\{\dot{S}(t)+S(t)A[q(t)]+A^T[q(t)]S(t)-S(t)BR^{-1}B^TS(t)+Q\}x(t)\le 0 \qquad (0\le t\le t_1, \forall q(t)\epsilon\Omega) \quad \text{(A2)}$$

Use of Eqs. (8) and (9) then yields

$$x^T(t)\{\dot{S}(t)+S(t)A[q(t)]+A^T[q(t)]S(t)+Q\}x(t)+x^T(t)S(t)Bu(t)+u^T(t)B^TS(t)x(t)+u^T(t)Ru(t)\le 0 \qquad (0\le t\le t_1, \forall q(t)\epsilon\Omega) \quad \text{(A3)}$$

Equation (1) yields

$$\frac{d}{dt}[x^T(t)S(t)x(t)] = x^T(t)\dot{S}(t)x(t)+x^T(t)S(t)\{A[q(t)]x(t)+Bu(t)\}+\{x^T(t)A^T[q(t)]+u^T(t)B^T\}S(t)x(t) \qquad (0\le t\le t_1) \quad \text{(A4)}$$

Combining Eqs. (A3) and (A4) yields

$$x^T(t)Qx(t)+u^T(t)Ru(t)\le -\frac{d}{dt}[x^T(t)S(t)x(t)] \qquad (0\le t\le t_1) \quad \text{(A5)}$$

Integration of Eq. (A5) from 0 to t_1, in conjunction with Eqs. (4) and (6), yields Eq. (10).

To prove the corollary to Theorem 1, note first that boundedness of J as $t_1\to\infty$ implies that the integrand in Eq. (4) tends to zero as $t_1\to\infty$. If q is constant, the observability and positivity assumptions which have been made guarantee that $x(t)$ tends to zero as $t\to\infty$, regardless of initial conditions, so that the closed-loop system is globally asymptotically stable.[18] If q is not constant, then the arguments are somewhat more complex. Details may be found in Refs. 14 and 26.

Proof of Theorem 2: The steady-state version of Eq. (14) may be written as

$$S_dA_0+A_0^TS_d-S_dBR^{-1}B^TS_d+Q+\rho_dU(S_d)=0 \quad \text{(A6)}$$

or as

$$S_dA(q)+A^T(q)S_d-S_dBR^{-1}B^TS_d+Q^*=0 \quad \text{(A7)}$$

where use has been made of Eqs. (3) and (17). Thus, the same matrix S_d may be viewed in two ways: it is the positive-definite steady-state solution to the guaranteed cost matrix Eqs. (6) and (14) for an imprecisely known system with dynamics matrix $A(q)$,

$$\rho_d a_i\le q_i\le\rho_d b_i \qquad (i=1,...,n') \quad \text{(A8)}$$

and state weighting matrix Q, and it is also the positive-definite steady-state solution to the Riccati equation for a precisely known system with dynamics matrix $A(q)$ and state weighting matrix $Q^*(q)$. Since the matrix S_d is positive-definite and satisfies an algebraic Riccati equation with a positive-definite control weighting matrix, a positive-semidefinite state weighting matrix, and controllability and observability conditions are satisfied, the controller described by Eqs. (15) and (16) is an optimal controller.

To prove the corollary to Theorem 2, note that

$$x^T\Big[\sum_{i=1}^{n'}q_i(S_dA_i+A_i^TS_d)\Big]x\le\rho x^TU(S_d)x \qquad (\rho\ge 0) \quad \text{(A9)}$$

for all x, if q is consistent with the inequalities

$$\rho a_i\le q_i\le\rho b_i \qquad (i=1,...,n') \quad \text{(A10)}$$

This follows from Eqs. (11-13) which define U. Thus, from Eqs. (17), (19), and (A9)

$$x^TQ^*(q)x\ge x^TQ_d(\rho)x \quad \text{(A11)}$$

for all x and all q consistent with Eq. (A10).

Thus, $Q^*(q)$ is guaranteed positive-semidefinite whenever $Q_d(\rho)$ is positive-semidefinite, which is for $\rho\le\rho_a$.

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